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A STUDY OF METHODS FOR THE DESIGN AND ANALYSIS OF SENSITIVITY EXPERIMENTS, QUARTERLY PROGRESS REPORT FOR PERIOD ENDING 30 JUNE 1964

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FOREWORD

This report was prepared by personnel of the Stochastic Processes and Operations Research Unit of the Rocketdyne Research Department under Contract NAS 8-11061, "A Study of Methods for the Design and Analysis of Sensitivity Experiments", for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of the Propulsion and Vehicle Engineering Division, Engineering Materials Branch of the Geor[redacted] C. Marshall Space Flight Center with Dr. John B. Gayle acting as [redacted] manager.

ABSTRACT

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This report summarizes work done under Contract NAS 8-11061 during the period 1 April 1964 to 30 June 1964. Included are discussions of recent work on the laboratory and literature surveys, design of experiment for the safety problem, coding of the simulation program, maximum likelihood estimates of partially ordered response probabilities, and simultaneous maximum likelihood estimates of location parameters for several sets of sesnitivity data.

Author

CONTENTS

SUMMARY

An algorithm has been found for computing maximum likelihood estimates of the response function in multivariate sensitivity experiments when monotonicity of the response function is hypothesized.

A program has been written to compute simultaneous maximum likelihood estimates of location parameters under the hypothesis that several response functions are non-decreasing and identical except for location, but otherwise unspecified.

The safety problem for a cumulative normal response function and a (locally) exponential stress density has been shown to require an experimental design which is asymptotic to that required for an associated inverse response problem. In this connection a result of Chernoff concerning the inverse response problem for a normal cdf response function has been independently derived.

Work has continued on the literature and laboratory surveys and on the program for simulating sensitivity experiments.

LABORATORY SURVEY

A report on the trip to the Washington area has been published (Ref. 1). Conclusions about the nature of sensitivity experiments derived from this trip (Ref. 2, p. 3) have been used to reorganize our efforts on this contract. A visit to Aerojet - Sacramento and UTC - Sunnyvale in July or August will conclude the travel required by the program. The following table indicates the present status of the laboratory survey.

Name of Organization	Administrator Handling Visit or Call	Status
Allegheny Ballistic Laboratory, Rocket Center, West Virginia	Mr. Robert H. Richardson, Supervisor, Sensitivity Group, Process Research Dept., Research Division	visited 2 April 1964
Bureau of Mines 4800 Forbes Avenue Pittsburgh 13, Pa.	Mr. Norm E. Hanna, Explosives Research Center	visited 23 March 1964
Aerojet-General Sacramento, Calif.	Mr. Todorov Dept. 4940	visit to be arranged
Aerojet-General Downey, Calif.	Mr. Donald E. Hartvigsen	mailed us methods of analysis 23 Dec. 1963
Battelle Memorial Institute, Columbus, Ohio	Mr. Arthur Levy, Physical Chemistry Section	no visit planned
Franklin Institute Laboratories for Research & Development, 20th St. Parkway, Philadelphia 3, Pa.	Mr. C. T. Davey Applied Physics Laboratory	visited 26 March 1964

Name of Organization	Administrator Handling Visit or Call	Status
Air Reduction Co. Murray Hill, New Jersey	Mr. Edson	no visit planned
Rohm and Haas Huntsville, Alabama	Dr. Frederic A. Johnson, Analytical Chemistry Group Redstone Arsenal Research Div.	visited 26 February 1964
Douglas Aircraft Corp., Santa Monica, Calif.	Mr. Andrew Economos	will send us bibliography and procedures
E. I. Du Pont de Nemours & Co. Wilmington 98, Delaware	Dr. Wendell Jackson, Assistant Manager, Research Explosives Dept.	no visit planned
Naval Ordnance Test Station, China Lake, Calif.	Mr. Jack Pakulak, Product Evaluation Branch 4532	visited 9 January 1964
Naval Ordnance Laboratory, White Oak, Silver Spring, Md.	Dr. Carl Boyars, Group WP	visited 31 March 1964
Jet Propulsion Lab., 4800 Oak Grove Drive, Pasadena, Calif.	Dr. Harold E. Marsh, Jr., Research Group Supervisor Solid Propellant Engineering	mailed procedures and sample data 13 January 1964
Naval Weapons Lab. Dahlgren, Virginia	Mr. Frank W. Kasdorf, Head, Terminal Effects Div.	visited 1 April 1964
U. S. Naval Propellant Plant Indian Head, Md.	Dr. Charles Dale, Explosive Safety Research Branch	visited 30 March 1964
Thiokol Chemical Corp. Brigham City, Utah	Explosive Safety Branch Wasatch Division	no contact yet
Picatinny Arsenal Dover, New Jersey	Dr. Harold J. Matsuguma, Chief, Explosives Laboratory	visited 25 March 1964

Name of Organization	Administrator Handling Visit or Call	Status
Frankford Arsenal Philadelphia 37, Pa.	Mr. Thomas Cicone, Chief, Combustion and Detonation Section	will mail information on request to Commanding Officer
Ballistic Research Labs. Aberdeen Proving Grounds, Md.	Mr. O. P. Bruno, Chief, Surveillance Group	visited 27 March 1964
Stanford Research Institute Menlo Park, Calif.	Mr. Lionel Dickenson	no visit planned
Lockheed Propulsion Co. Box 111 Redlands, Calif.	Mr. A. T. Camp, Director, Propellant Development Div.	mailed procedures 5 February 1964
United Technology Corp. Sunnyvale, Calif.	Dr. Bernard L. Iwanciow, Ballistics Section, Research & Advanced Technology Dept.	visit to be arranged
Aeronutronic Newport Beach, Calif.	Mr. H. J. Langlie, Manager, Reliability Dept.	visited 9 March 1964
Space and Information Systems Div., North American Aviation, Inc.	Dr. C. N. Scully, and Mr. Alfred Africano	visited 9 March 1964
National Bureau of Standards, Washington, D. C.	Dr. Churchill Eisenhart, Code 154	telephoned 30 March 1964
Los Alamos Scientific Laboratory, Los Alamos, New Mexico	Dr. R. Keith Zeigler, T1 Statistical Section	telephoned
Harvard University Cambridge, Mass.	Professor Cochran, Statistics Department	no contact yet
UCLA Los Angeles, Calif.	Prof. W. J. Dixon	indirect contact

Name of Organization	Administrator Handling Visit or Call	Status
Lawrence Radiation Lab. Livermore, Calif.	Dr. G. Dorough, Chemistry Division	no visit planned
Armour Research Foundation, Illinois Institute of Technology	T. A. Erikson, or E. L. Grove	no contact yet

LITERATURE SURVEY

In the course of the literature survey we have examined the work of Katz (Ref. 3) on estimating a pair of ordered probabilities from two equal-sized samples as a possible basis for an alternative procedure to the method of reversals. Although the extension to unequal sample sizes would not be difficult, the extension to three or more ordered probabilities would be so much more complex than the (maximum likelihood) method of reversals that we do not now plan to investigate the technique any further.

A paper by Chernoff (Ref. 4) was found to contain results which we reported in the last monthly letter as original; however, our method of derivation was different (see below, The Safety Problem).

ANALYSIS OF SENSITIVITY DATA

THE MINIMUM OVERLAPPING SUBSET

The computation of the minimum overlapping subset (MOS) of a set of sensitivity data (see Ref. 2, pp. 13-15) has been programmed (in FORTRAN) for digital computation as an independent program and also as a subroutine. Approximate estimates of the mean and standard deviation of the critical level density provided by the MOS have also been programmed. These approximations continue to perform well relative to the exact maximum likelihood estimators which are far more laborious to compute.

LOCATION PARAMETERS

Suppose that we have several different explosives or batches of the same explosive supplied at different times, and we wish to rank them according to sensitivity after collecting data on each. If the experimentally applied stresses do not correspond to the real stress variable in actual use, or if the explosive samples do not correspond to the actual charges to be used, then we will not be able to rank the explosives strictly according to safety. However, in some cases it may be sufficient to perform a ranking which is independent of stress density. One solution to this problem is to imagine that the response functions for the different explosives are identical except for location. Then the location parameters for all of the different explosives would be estimated simultaneously and the estimates used to establish a sensitivity scale (as suggested in Ref. 2, p. 3, classification 3).

One (parametric) approach to this problem would be to assume that the response functions are normal cumulative distribution functions (cdf's) with a common standard deviation. Maximum likelihood estimates of the common standard deviation and the different means would then be computed through a Newton-Raphson iterative procedure, using a modification of the minimum overlapping subset technique to provide first guess of the estimates. These computations will be programmed in the next quarter; no particular difficulty is expected.

A non-parametric approach would be to assume that the response functions are non-decreasing and identical except for location, but are otherwise unspecified. In this case (which we denote by H_o, the null hypothesis) the problem of maximum likelihood estimation is complicated by two situations illustrated in the following canonical pathology:

Set Number	Response Fraction at $x = 0$	Response Fraction at $x = 1$
1	0/1	1/1
2	1/1	0/1
3	1/4	
4		3/4

Let δ_i denote the location parameter for the i^{th} population, and set $\delta_1 = 0$. Let L_{12} denote the joint likelihood of the first two sets of data. Then

$$L_{12} = \begin{cases} 4/27 & \text{for} \quad \delta_2 < 0 \\ 1/16 & \text{for} \quad \delta_2 = 0 \\ 4/27 & \text{for} \quad \delta_2 > 0 \end{cases}$$

Thus the likelihood function is not unimodal in the location parameters, and in this example isn't even uniquely maximized.

Furthermore, if we choose $\hat{\delta}_2 < 0$, where $\hat{\delta}$ denotes a maximum likelihood estimate of δ based on all of the data, then we would find $\hat{\delta}_4 < \hat{\delta}_2 < 0$, whereas, if we deleted set number 2, we should have $\hat{\delta}_4 > 0$. And if we choose $\hat{\delta}_2 > 0$, then we would find $\hat{\delta}_3 > \hat{\delta}_2 > 0$, whereas, if we deleted set number 2, we would have $\hat{\delta}_3 < 0$. Thus, relative location estimates are functions of all the data, and not just the two sets in question.

One iterative technique for obtaining maximum likelihood estimates in such problems that was considered used as initial guesses for the location parameters the centers of the open interval solutions to the 50% inverse response problem given by the method of reversals applied independently to each set of data. We have now found that this is unsatisfactory and have replaced it by first guesses supplied by the minimum overlapping subset algorithm. These were applied to seventeen sets of percussion primer data supplied by Ballistic Research Labs, Aberdeen Proving Grounds. A preliminary analysis of the data indicated that it would appear reasonable to accept the null hypothesis, H_o, if a logarithmic transformation was applied to the stress variable. Subsequently, the transformed data was analyzed by a simple program which has been written to search for a local maximum of the joint sample likelihood function. The output of this program for the BRL data is summarized below. If the "estimated location parameter" values are subtracted from the "log (stress)" values, the resulting data for the seventeen batches do indeed appear to come from the same population. Unfortunately, there is still no guarantee that this local solution is actually a global optimum. On the other hand, this particular example is unusually complex because of the number of batches of data; the program would be more trustworthy for smaller problems, since in this case the local optimization is more likely to provide a global solution.

In order to test H_o against H_1, the alternative hypothesis that the response functions are non-decreasing but otherwise unrelated, it was decided to use a χ^2 test. Unfortunately, it is not obvious what to use for the "degrees of freedom" parameter because estimation by the method of reversals does not clearly correspond to a particular integer number of degrees of freedom (compared to the two degrees of freedom, for example, which are "lost" when performing straight-line regression). Further work on this problem is necessary.

PERCUSSION PRIMER SENSITIVITY DATA SUPPLIED BY BALLISTIC RESEARCH LABS

LOG (Stress)	Responses	Trials	Estimated Location Parameter
1.10	0	1	1.429
1.18	0	1	
1.25	0	2	
1.32	1	7	
1.39	5	15	
1.45	9	14	
1.50	4	7	
1.56	3	3	
1.01	0	1	1.281
1.10	1	5	
1.18	4	8	
1.25	4	13	
1.32	9	19	
1.39	11	14	
1.45	4	5	
1.50	1	1	
.92	0	2	1.164
1.01	2	9	
1.10	7	16	
1.18	9	16	
1.25	8	13	
1.32	6	6	
1.39	1	2	
1.45	2	2	
1.01	0	1	1.266
1.10	1	5	
1.18	4	10	
1.25	6	17	
1.32	11	18	
1.39	8	11	
1.45	4	4	

LOG (Stress)	Responses	Trials	Estimated Location Parameter
1.10	0	5	1.254
1.18	5	16	
1.25	11	23	
1.32	12	16	
1.39	5	5	
1.45	1	1	
1.10	0	5	1.266
1.18	5	16	
1.25	12	19	
1.32	8	14	
1.39	7	9	
1.45	3	3	
1.01	0	1	1.260
1.10	1	5	
1.18	4	11	
1.25	8	20	
1.32	13	18	
1.39	6	8	
1.45	3	3	
.81	0	6	1.008
.92	6	16	
1.01	10	20	
1.10	11	13	
1.18	3	5	
1.25	3	3	
1.32	1	1	
1.39	1	1	
1.45	1	1	

PERCUSSION PRIMER SENSITIVITY DATA SUPPLIED BY BALLISTIC RESEARCH LABS

LOG (Stress)	Responses	Trials	Estimated Location Parameter
.81	0	2	1.082
.92	2	8	
1.01	6	14	
1.10	9	17	
1.18	9	14	
1.25	6	7	
1.32	2	2	
1.39	1	1	
1.45	1	1	
1.10	0	1	1.392
1.18	1	4	
1.25	3	7	
1.32	4	12	
1.39	8	14	
1.45	6	14	
1.50	7	11	
1.56	3	3	
1.10	0	5	1.281
1.18	5	12	
1.25	7	17	
1.32	10	16	
1.39	7	10	
1.45	4	5	
1.50	1	1	
1.18	0	1	1.344
1.25	1	12	
1.32	11	23	
1.39	13	21	
1.45	9	9	
1.10	0	2	1.302
1.18	2	9	
1.25	7	17	
1.32	10	18	
1.39	9	14	
1.45	6	6	

LOG (Stress)	Responses	Trials	Estimated Location Parameter
1.18	0	2	1.356
1.25	2	12	
1.32	10	19	
1.39	9	18	
1.45	9	12	
1.50	3	3	
1.39	0	1	1.650
1.45	1	2	
1.50	0	4	
1.56	3	11	
1.61	7	16	
1.66	8	12	
1.70	3	10	
1.75	6	8	
1.79	1	2	
1.25	0	8	1.385
1.32	8	16	
1.39	8	19	
1.45	11	15	
1.50	4	6	
1.56	2	2	
1.45	0	3	1.623
1.50	2	10	
1.56	7	16	
1.61	8	13	
1.66	4	10	
1.70	5	7	
1.75	1	3	
1.79	2	3	
1.83	1	1	

AN ALGORITHM FOR COMPUTING MAXIMUM LIKELIHOOD
ESTIMATES IN MULTIVARIATE SENSITIVITY EXPERIMENTS

The method of reversals (Ref. 5) provides an algorithm for the computation of maximum likelihood estimates for univariate sensitivity experiments in which the probability of response is assumed monotone nondecreasing with increasing stress. A more complex algorithm has been developed in this quarter for obtaining analogous maximum likelihood estimates in multivariate sensitivity experiments when a monotonicity hypothesis (naturally generalized from the univariate case) of the form

$$[x_i \leq y_i,\ i = 1,\ \ldots,\ k] \Rightarrow [M(x_1,\ \ldots,\ x_k) \leq (y_1,\ \ldots,\ y_k)]$$

is assumed (see Ref. 2 pp. 11-13). Here $M(x_1, \ldots, x_k)$ denotes the response probability for the values $x_1, \ldots, x_k$ of the k stimulus variables.

In univariate experiments the natural ordering of the stimulus variable permits implementation of the one-dimensional version of the above monotonicity hypothesis and this leads to a (trivial) unique ordering of the responses recorded at the various stimuli. In the multivariate case, however, only a partial ordering of the stimuli is available. For example, let (x_1, y_1) and (x_2, y_2) be two distinct stimulus-level combinations for the stimuli x and y. If $x_1 < x_2$, $y_1 \leq y_2$ or $x_1 \leq x_2$, $y_1 < y_2$, the order (x_1, y_1), (x_2, y_2) of increasing stimulus is the "natural" one to use in evaluating the responses at these combinations.

If, on the other hand, neither of these inequalities holds, then no such natural ordering can be established.

The new algorithm developed makes use of the natural order together with the observed fraction responses to obtain a complete ordering of the response probabilities. When the method of reversals is applied to the results of this procedure, maximum likelihood estimates of the response probabilities are obtained.

The procedure can be described as follows (for simplicity of notation we use the two variable case):

Consider two points (x_1, y_1) and (x_2, y_2) which are not in natural order; suppose the observed proportions of responses are r_1 and r_2, respectively, at these two points. If there are no constraints on the estimates then the maximum likelihood estimates of the probabilities at the two points are $\hat{M}(x_1, y_1) = r_1$ and $\hat{M}(x_2, y_2) = r_2$, respectively. The points (x_1, y_1) and (x_2, y_2) are then ordered in terms of increasing probability estimates; e.g., the ordering would be (x_1, y_1), (x_2, y_2) if $r_1 \leq r_2$. This ordering procedure will be referred to as a <u>minimum ascent order</u>.

Consider now a sensitivity experiment with k stimulus variables. The following definitions are useful in expressing the conditions for a minimum ascent ordering.

1) Natural precedence. If $R^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_2^{(1)}, \ldots, x_k^{(1)})$ and $R^{(2)} = (x_1^{(2)}, x_2^{(2)}, \ldots, x_k^{(2)})$ are two stimulus-level combinations, and if $x_i^{(1)} \leq x_i^{(2)}$ $(i = 1, 2, \ldots, k)$ and for one of these indices j, $x_j^{(1)} < x_j^{(2)}$, then $R^{(1)}$ will be said to have natural precedence over $R^{(2)}$. This will be indicated by $R^{(1)} << R^{(2)}$.

2) Complete set. Consider a set S of stimulus level combinations. Let $R^{(1)}$ and $R^{(3)}$ be any two elements of S. If for any $R^{(2)}$ satisfying $R^{(1)} << R^{(2)} << R^{(3)}$ it follows that $R^{(2)} \in S$, then we will say that S is a complete set.

3) Independence. If S_1 and S_2 are complete sets and no element of one has natural precedence over any element of the other, then S_1 and S_2 will be called independent.

4) Associated probability. Suppose the set $S = \{R^{(i)} | i = 1, 2, \ldots, k\}$ is complete. If in n_i tests at $R^{(i)}$, f_i failures are observed, then with S we will associate the probability estimate, $\hat{P}_S$, given by

$$\hat{P}_S = \frac{f_1+f_2+\ldots,+f_k}{n_1+n_2+\ldots,+n_k}$$

In terms of these definitions a minimum ascent ordering may be defined precisely by the following two conditions:

A) If $R^{(1)} << R^{(2)}$, then $R^{(1)}$ will precede $R^{(2)}$ in a minimum ascent ordering. It follows directly that if a minimum ascent ordering is given by

$$R^{(1)}, R^{(2)}, R^{(3)}, \ldots, R^{(j)}, R^{(j+1)}, \ldots, R^{(j+\ell)}, \ldots, R^{(k)}. \quad (1)$$

then for any $j \geq 1$ and $j + \ell \leq k$, the subset

$$S = \{R^{(i)} | i = j, j+1, \ldots, j+\ell\} \text{ is a complete set.}$$

B) If (1) is a minimum ascent ordering, and for any m $(0 < m < \ell)$ the sets $S_1 = \{R^{(i)} | i = j, j+1, \ldots, j+m\}$ and $S_2 = \{R^{(i)} | i = j+m+1, \ldots, j+\ell\}$ are independent, then the associated probabilities satisfy the relation $\hat{P}_{S_1} \leq \hat{P}_{S_2}$.

It can be shown that in the case $k = 2$, these conditions yield the heuristic definition of a minimum ascent ordering given earlier.

In order to obtain maximum likelihood estimates under monotonicity in the multivariate case we write the observed individual response functions in an order that satisfied the conditions A and B above, then find a maximum likelihood partition by the method of reversals as derived in Ref. 5, and then find a "best" minimum ascent ordering.

In this connection we have proven the following two theorems (the proofs are given in Ref. 6).

<u>Theorem 1</u>. Consider the class, C, of all orderings which are consistent with the natural order. In this class there is a subclass of orderings which contains at least one minimum ascent ordering whose maximum

likelihood partition results in maximum likelihood estimates relative to all orderings in C.

Theorem 2. All minimum ascent orderings result in the same maximum likelihood partition.

Although there is not a unique minimum ascent ordering, we see by Theorem 2 that all of them have the same maximum likelihood partition and therefore result in the same response-fraction estimates. From Theorem 1, we have that for one minimum ascent ordering its maximum likelihood partition gives maximum likelihood estimates under the required constraints, and thus any minimum ascent ordering does.

In order to actually generate a minimum ascent ordering the stimulus levels can be written in an order in which condition (A) is satisfied. (The simplest is to order in terms of one stimulus at a time.) Then, independent sets are reordered until (B) is satisfied. This procedure can be best illustrated by an example.

Suppose sensitivity tests are conducted at sixteen points arranged in a 4 x 4 grid with the proportions of response at these points as follows:

j = 4	1/4	1/3	1/2	1/3
j = 3	1/3	1/4	1/5	1/4
j = 2	1/5	1/8	1/6	1/4
j = 1	1/3	1/4	1/8	1/6
	i=1	i=2	i=3	i=4

By appealing to natural precedence we may derive the following trial ordering of the points R_{ij}.

$R_{11}(1/3)$ $R_{12}(1/5)$ $R_{13}(1/3)$ $R_{14}(1/4)$ | $R_{21}(1/4)$ $R_{22}(1/8)$ $R_{23}(1/4)$ $R_{24}(1/3)$ |
$R_{31}(1/8)$ $R_{32}(1/6)$ $R_{33}(1/5)$ $R_{34}(1/2)$ | $R_{41}(1/6)$ $R_{42}(1/4)$ $R_{43}(1/4)$ $R_{44}(1/3)$

where the vertical lines indicate breaks in the natural order. For convenience the data are given in parenthesis after each point. Consider the pair of independent complete sets $S_1 = \{R_{13}, R_{14}\}$ and $S_2 = \{R_{21}, R_{22}\}$. The associated probability estimates are $\hat{P}_{S_1} = 2/7$ and $\hat{P}_{S_2} = 1/6$. The order of these sets should therefore be reversed since the former estimate is the larger. Similarly, the following pairs of sets should be reversed: $\{R_{23}, R_{24}\}$ and $\{R_{31}, R_{32}\}$, $\{R_{33}, R_{34}\}$ and $\{R_{41}\}$. The overall order now has become

$R_{11}(1/3)$ $R_{12}(1/5)$ | $R_{21}(1/4)$ $R_{22}(1/8)$ | $R_{13}(1/3)$ $R_{14}(1/4)$ |
$R_{31}(1/8)$ $R_{32}(1/6)$ | $R_{23}(1/4)$ $R_{24}(1/3)$ | $R_{41}(1/6)$ | $R_{33}(1/5)$ $R_{34}(1/2)$ |
$R_{42}(1/4)$ $R_{43}(1/4)$ $R_{44}(1/3)$.

We now note that the following pairs should be reversed

$$\{R_{13}, R_{14}\} \text{ and } \{R_{31}, R_{32}\},$$
$$\{R_{41}\} \text{ and } \{R_{23}, R_{24}\},$$
$$\{R_{34}\} \text{ and } \{R_{42}, R_{43}\},$$

giving the order

$$R_{11}(1/3)\ R_{12}(1/5)\ |\ R_{21}(1/4)\ R_{22}(1/8)\ |\ R_{31}(1/8)\ R_{32}(1/6)\ |$$
$$R_{13}(1/3)\ R_{14}(1/4)\ |\ R_{41}(1/6)\ |\ R_{23}(1/4)\ R_{24}(1/3)\ |\ R_{33}(1/5)\ |$$
$$R_{42}(1/4)\ R_{43}(1/4)\ |\ R_{34}(1/2)\ R_{44}(1/3).$$

There remain the reorderings

$$\{R_{13}, R_{14}\} \text{ and } \{R_{41}\}$$
$$\{R_{24}\} \text{ and } \{R_{33}, R_{42}, R_{43}\}$$

which results in the final order

$$R_{11}(1/3)\ R_{12}(1/5)\ |\ R_{21}(1/4)\ R_{22}(1/8)\ |\ R_{31}(1/8)\ R_{32}(1/6)\ |\ R_{41}(1/6)\ |$$
$$R_{13}(1/3)\ R_{14}(1/4)\ |\ R_{23}(1/4)\ R_{33}(1/5)\ |\ R_{42}(1/4)\ R_{43}(1/4)\ |$$
$$R_{24}(1/3)\ R_{34}(1/2)\ R_{44}(1/3)\ ,$$

which is a minimum ascent order, as can be verified by inspection. We can now partition our minimum ascent order as follows:

$$\underset{x_1}{R_{11}\ R_{12}\ R_{21}\ R_{22}\ R_{31}\ R_{32}\ R_{41}}\ |\ \underset{x_2}{R_{13}\ R_{14}\ R_{23}\ R_{33}\ R_{42}\ R_{43}}\ |$$
$$\underset{x_3}{R_{24}}\ |\ \underset{x_4}{R_{34}\ R_{44}}.$$

The probability estimates which arise from this partition are

$$P_{x_1} = \frac{7}{40} \quad P_{x_2} = \frac{6}{24} \quad P_{x_3} = \frac{1}{3} \quad P_{x_4} = \frac{2}{5}$$

and our maximum likelihood partition divides the responses as follows:

1/4	1/3	1/2	1/3
1/3	1/4	1/5	1/4
1/5	1/8	1/6	1/4
1/3	1/4	1/8	1/6

SIMULATION OF SENSITIVITY EXPERIMENTS

The FORTRAN program for simulating sensitivity experiments (Ref. 2, pp. 20-21) has been checked out, and sample outputs have been forwarded to the contract monitor for comment. The cases run are for the Bruceton up-and-down design, with samples up to size 1024, when the response function is a normal cdf with parameters μ and σ, the initial test is at μ and the test level increment is $.6745\ \sigma$. One of the printed outputs is reproduced below (CRT output is also available).

At a conference with the contract monitor several suggestions were made for improving the computer program, such as printing out the theoretical value of the covariance matrix of the estimates, when it is known. The program is now being modified to incorporate this and other suggestions.

EXPERIMENTAL DESIGN - BRUCETON UP-AND-DOWN, INITIAL TEST LEVEL = 9, TEST LEVEL INCREMENT = 1.0
ANALYTICAL PROCEDURE - DIXON AND MASSEY
RESPONSE FUNCTION - CUMULATIVE NORMAL, MEAN = 9, STANDARD DEVIATION = 1.4826
NUMBER OF SIMULATIONS - 100

SAMPLE SIZE 1024

	PARAMETER 1 (MU)	PARAMETER 2 (SIGMA)
TRUE VALUE	9.0000E 00	1.4826E 00
SAMPLE MEDIAN OF THE ESTIMATES	8.9956E 00	1.4830E 00
SAMPLE MEAN OF THE ESTIMATES	8.9979E 00	1.4792E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.0690E-02	1.0528E-01
SMALLEST ESTIMATE	8.8679E 00	1.2239E 00
LARGEST ESTIMATE	9.1152E 00	1.7340E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.6833E-03	-7.1724E-05
-7.1724E-05	1.1084E-02

THE SAFETY PROBLEM

In many sensitivity experiments, primary interest centers about the safety of a material or process, which may be defined as the probability that the system will survive a certain range of stresses involved in shipping, storage, etc. If $g(x)$ denotes the density of maximum stresses encountered by individual systems, and if $M(x)$ denotes the response function, which is the probability of response as a function of stress, then the safety may be defined formally by

$$S \equiv 1 - \int_{-\infty}^{\infty} g(x)M(x)dx \; .$$

During the past quarter we have examined the case when $M(x)$ is the normal cdf with parameters μ and σ and where $g(x)$ can be represented by Ae^{-Bx} in the range of interest (e.g., $x > \mu-4\sigma$). Then, integrating by parts, we have

$$1 - S \cong \int_{-\infty}^{\infty} Ae^{-Bx} \int_{-\infty}^{x} \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\mu)^2/2\sigma^2} dy\right) dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{A}{B} e^{-Bx}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}\right) dx$$

$$= \frac{A}{B\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu+B\sigma^2)^2/2\sigma^2} \, e^{-\mu B+B^2\sigma^2/2} \, dx$$

$$= \frac{A}{B} e^{-B(\mu-B\sigma^2/2)} \quad .$$

If the sole purpose of the sensitivity experiment is to determine the safety of the system, then only the function $f = \mu - B\sigma^2/2$ need be estimated, and not the individual parameters. Now the variance of the estimate of f is given approximately by VQV^T, where $V = (\partial f/\partial\mu, \partial f/\partial\sigma)$, Q denotes the covariance matrix of the estimates of the individual parameters, and V^T denotes the transpose of the vector V. Since $\partial f/\partial\mu = 1$ and $\partial f/\partial\sigma = -B\sigma \equiv K$, a test design which is optimal for the above problem, in the sense of minimizing the variance of the estimate of f, will be asymptotic to that design which is optimal for estimating $\mu+K\sigma$. This is a parametric version of the inverse response problem.

An asymptotically optimal design for estimating $\mu+K\sigma$ was first found by Chernoff (Ref. 4) and has now been derived independently by us. Briefly, the solution is divided into two cases:

Case I: $|K| \geq 1.5750360$

In this case $N(.5 + .787518/K)$ of the (N) tests should be at $\mu + 1.5750360\sigma$, and $N(.5 - .787518/K)$ of the tests should be at $\mu - 1.5750360\sigma$. The variance of the estimate is asymptotic to $1.6436K^2\sigma^2/N$*, compared to $(2.5 + 3.2K^2)\sigma^2/N$ for the Langlie design (Ref. 7) and to $(2.0 + 3.8K^2)\sigma^2/N$ for the Bruceton design with test

*The factor $(1+K^2)^{-1}$ is incorrectly included in the equations at the bottom of p. 9, Ref. 4.

level increment equal to the standard deviation (Ref. 8).

Case II: $|K| < 1.5750360$

All of the tests should be at $\mu+K\sigma$. Even at $K = 0$, where the two commonly used alternatives perform relatively well, the variance of the estimate for the asymptotically optimal design is asymptotic to $(\pi/2)\sigma^2/N$, whereas it is $2.5\sigma^2/N$ for the Langlie design and $1.85\sigma^2/N$ for the Bruceton design when the test level increment is $.67\sigma$, a relatively favorable choice of spacing.

Thus, in both cases it has been concluded that the Bruceton and Langlie designs are simply too inefficient to be recommended without reservation for the inverse response problem, although they may still turn out to be useful for very small samples.

Actually, the design recommended above cannot be exactly followed since μ and σ are not known and hence the recommended test level(s) is not known exactly. But since the present solution is only asymptotically efficient, this property is preserved if the above recommendation is implemented with designs which merely converge to the optimum test level(s) with probability one. This problem is more complex when, as in the safety problem being considered, K itself is a function of μ and/or σ, but we do not anticipate much difficulty in finding some sort of convergent procedure.

A memo on these results will be published shortly. It is planned to compare the three designs discussed above for the small sample case by means of the simulation program described earlier.

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